Int. J. Solids Structures, 1972, Vol. 8, pp. 807 to 823. Pergamon Press. Printed in Great Britain

GENERALIZED TORSIONAL WAVES AND THE NON-AXISYMMETRIC END PROBLEM IN A SOLID CIRCULAR CYLINDER

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Abstract—The dispersion relations obtained from the approximate dynamical theory of torsion developed in [1] for the problem of plane waves propagating in an infinite circular cylinder are compared with the corresponding relations of the exact three-dimensional theory and are found to be in close agreement. End effects in static torsion are investigated and the solution to a problem of the non-axisymmetric torsion of a circular rod is obtained. It is found that the rate of decay of non-axisymmetric end effects is slower than that of the analogous axisymmetric end effects.

1. INTRODUCTION

IN A previous paper [1] a dynamical theory of torsion was derived which included the effects of the warping and in-plane shearing motions that, in general, accompany torsional deformations in cylindrical rods. This approximate theory is governed by a system of coupled one-dimensional equations in three displacement functions, two of which combine to describe the torsional and in-plane shearing motion (which we call contour-shear motion) while the third describes the warping motion. In [1] the problem of plane waves propagating in an infinite rod was solved within the context of this theory. The details of the solution depend upon the material properties of the rod, the geometry of the cross section and the so-called "correction factors". In [1] these correction factors were chosen so that the values of the contour-shear and warping cut-off frequencies together with the torsional rigidity and the asymptotic phase and group velocity obtained from the approximate theory were identical to those obtained from the exact three-dimensional theory of elasticity. For a rod of circular cross section the correction factors were evaluated explicitly and dispersion relations were obtained.

In the present paper a detailed comparison is made between the dispersion relations of this approximate solution and the dispersion relations of the exact three-dimensional solution. This comparison establishes the range of applicability of the approximate theory. It is found that the two sets of dispersion relations are in good qualitative and quantitative agreement up to relatively high frequencies. At these higher frequencies, as might be expected, modes of the exact solution not included in the present theory become of importance and the coupling of these modes to the modes which have been included causes portions of the two sets of curves to diverge.

Near zero frequency the two sets of curves (which do not coincide exactly) yield wave-numbers which correspond to displacements confined near the ends of a rod. Problems involving these types of displacements have been treated within the framework of the exact theory where they are referred to as "end problems". Some information concerning this class of problems has recently been obtained through a renewed interest in the "Saint-Venant principle of equipollent loads" with particular emphasis on energy considerations (see e.g. [2]). Explicit solutions of these problems on the other hand have proven quite difficult to obtain, even for the case of a circular cylinder. For example, the Saint-Venant solution for static torsion of a circular cylinder was amended by Purser [3] to account for end effects in the general case of axisymmetric torsion, however the more general problem of non-axisymmetric torsion has apparently not been solved in the three-dimensional theory.

To indicate how these problems may be handled in the context of the approximate theory, a problem of the non-axisymmetric torsion of a circular rod is solved and for comparative purposes, the problem of axisymmetric torsion is also solved. As mentioned previously, the two sets of dispersion relations do not coincide at zero frequency, but in order to obtain more accurate numerical results with the approximate theory for this class of problems exact matching is desirable. This match is obtained by an appropriate alternate choice of correction factors. Employing the new set of correction factors it is found that the solution of the non-axisymmetric problem consists of the solution of the axisymmetric problem plus solutions which decay exponentially from the ends of the rod. Thus, with distance from either end the solution of the non-axisymmetric problem approaches that of the axisymmetric problem. The crucial feature of these solutions is the exponential decay, which is determined by the imaginary part of the wavenumbers at zero frequency. The smaller this imaginary part is, the more slowly the solutions decay. Though the non-axisymmetric problem has not been solved in the exact theory, the type and character of the solutions involved have been obtained by Dougall [4] (see also [5]) and are available for comparison. It is shown that the minimum decay which might be expected from these solutions of the exact theory corresponds precisely to the decay as predicted from the approximate theory.

2. PLANE WAVES IN AN INFINITE CIRCULAR CYLINDER

The coupling between torsional, contour-shear and warping modes of motion is taken into account in the one-dimensional equations governing the torsional motions of an elastic rod derived in Ref. [1]. Referred to a rectangular cartesian coordinate system with x_3 denoting the axis of the rod, x_1 and x_2 denoting axes in the plane of the cross section and employing the notations of a superposed dot indicating differentiation with respect to time, t, and a comma followed by an index indicating differentiation with respect to the corresponding spatial coordinate these equations consist of *the stress-moment equations of motion*:

$$T^{(0,1)}_{31,3} - T^{(0,0)}_{12} = \rho I^*_{02}\ddot{u}^{(0,1)}_1,$$

$$T^{(1,0)}_{32,3} - T^{(0,0)}_{12} = \rho I^*_{20}\ddot{u}^{(1,0)}_2, \tag{1}$$

$$T^{(1,1)}_{33,3} - T^{(0,1)}_{31} - T^{(1,0)}_{32} = \rho I^*_{22}\ddot{u}^{(1,1)}_3$$

and the constitutive equations:

$$
\begin{aligned}
T_{12}^{(0,0)} &= \mu I_{00}^{*}(u_2^{(1,0)} + u_1^{(0,1)}),\\
T_{31}^{(0,1)} &= \mu I_{02}^{*}(u_{1,3}^{(0,1)} + u_3^{(1,1)}),\\
T_{32}^{(1,0)} &= \mu I_{20}^{*}(u_{2,3}^{(1,0)} + u_3^{(1,1)}),\\
T_{33}^{(1,1)} &= E I_{22}^{*} u_{3,3}^{(1,1)},
\end{aligned} \tag{2}
$$

where $E = [\mu(3\lambda+2\mu)]/(\lambda+\mu)$ is Young's modulus, λ and μ are the Lamé constants and ρ is the density of the rod. In equations (1) and (2) $T_{12}^{(0,0)}$, $T_{31}^{(0,1)}$, $T_{32}^{(1,0)}$ and $T_{33}^{(1,1)}$ are stress-moments, $u_1^{(0,1)}$, $u_2^{(1,0)}$ and $u_3^{(1,1)}$ are generalized torsional displacements and $I_{00}^{*}(\equiv \kappa_{00}I_{00})$, $I_{02}^{*}(\equiv \kappa_{02}I_{02})$, $I_{20}^{*}(\equiv \kappa_{20}I_{20})$ and $I_{22}^{*}(\equiv \kappa_{22}I_{22})$ are corrected moments of the area of the cross section where I_{00}, I_{02}, I_{20} and I_{22} are moments of the area and $\kappa_{00}, \kappa_{02}, \kappa_{20}$ and κ_{22} are the correction factors. All of these quantities are defined as in Ref. [1].

In [1] a solution of equations (1) and (2) was found for the problem of plane waves propagating in an infinite circular rod of radius a. Employing the dimensionless frequency $\Omega \equiv \omega a/v_s$ and dimensionless wavenumber $\phi \equiv \xi a$ where ω is the angular frequency, $v_s = (\mu/\rho)^{1/2}$ is the shear wave velocity and ξ is the wavenumber, the dispersion relations of this solution can be written in the form

$$\Omega^2 - \phi^2 = 0, \tag{3}$$

$$[\Omega^2 - 8\kappa_{00} - \phi^2][\Omega^2\kappa_{22}/12 - (1+\nu)\kappa_{22}\phi^2/6 - 1] - \phi^2 = 0, \tag{4}$$

where ν is Poisson's ratio. Equation (3) represents the axisymmetric torsional branch and equation (4) represents the non-axisymmetric contour-shear and warping branches of the approximate theory. The correction factors κ_{00} and κ_{22} were chosen in [1] so that the contour-shear and warping cut-off frequencies as determined by (4) are identical to those obtained from the exact three-dimensional theory. In order to assess the region of applicability of the solutions (3) and (4) we now compare these results with solutions of the exact theory.

Within the framework of the exact three-dimensional theory the frequency equation for plane waves propagating in an infinite circular rod was obtained by Hudson [6]. This frequency equation depends upon the number of nodal diameters n in the plane of the cross section and in the present notation can be written,

$$F_n(\Omega^2, \phi^2) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \tag{5}$$

where

$$
\begin{aligned}
a_{11} &= 2n(xJ_n'(x) - J_n(x)),\\
a_{12} &= 2(-xJ_n'(x) - x^2 J_n(x) + n^2 J_n(x)),\\
a_{13} &= -2yJ_n'(y) + 2n^2 J_n(y) - x^2 J_n(y) + \phi^2 J_n(y),\\
a_{21} &= 2xJ_n'(x) + x^2 J_n(x) - 2n^2 J_n(x),\\
a_{22} &= 2n(J_n(x) - xJ_n'(x)),\\
a_{23} &= 2n(J_n(y) - yJ_n'(y)),\\
a_{31} &= n\phi^2 J_n(x),\\
a_{32} &= x(\phi^2 - x^2)J_n'(x),\\
a_{33} &= 2\phi^2 yJ_n'(y),
\end{aligned} \tag{6}
$$

where

$$x^2 = \Omega^2 - \phi^2, y^2 = h^{-2}\Omega^2 - \phi^2, h^2 = \frac{2(1-\nu)}{1-2\nu} \tag{7}$$

and J_n is the Bessel function of the first kind of order n and the prime indicates differentiation with respect to the argument.

For each $n \geq 1$ a distinct frequency equation results from (5) and for $n = 0$ two distinct frequency equations result. The frequency equation for the family of flexural modes is determined by taking $n = 1$ and the frequency equations for the so-called "flexural branches of higher circumferential order" are determined by taking $n \geq 2$ [7]. For the special case $n = 0$ equation (5) yields the two independent frequency equations,

$$a_{21} = 0 \quad \text{and} \quad a_{12}a_{33} - a_{13}a_{32} = 0, \tag{8}$$

which govern the axisymmetric families of torsional and extensional modes, respectively.

With the aid of recurrence relations for Bessel functions the frequency equation for the family of torsional modes $a_{21} = 0$ can be written in the simple form,

$$J_2(x) = 0. \tag{9}$$

These modes are characterized by a single angular displacement which is axisymmetric. The lowest torsional branch is given by $x = 0$, the smallest root of (9) and in view of the definition of x^2 [see equation (7)], this branch is identical to that of the approximate theory [see equation (3)]. The next higher root of (9) is $x = 5{\cdot}1356$ and the frequency equation for this branch is $(5{\cdot}1356)^2 = \Omega^2 - \phi^2$. This branch has a cut-off frequency $\Omega_2 = 5{\cdot}1356$ and zero frequency intercepts given by $\phi = \pm j(5{\cdot}1356)$ where $j = (-1)^{\frac{1}{2}}$.

The contour-shear and warping branches of the exact solution are the two lowest branches of the frequency equation (5) obtained by taking $n = 2$. For this case the determinantal equation (5) is a transcendental equation in ϕ^2 and Ω^2. For a given Ω the wave-numbers satisfying (5) may be real, imaginary or complex. Moreover, if for a given Ω, ϕ is a solution of (5) then $-\phi$ and the complex conjugates of ϕ and $-\phi$ are also solutions. Zemanek [8] has investigated several of the branches of (5) for various values of n. For $n = 2$ he has investigated roots of (5) for real and imaginary wavenumbers and he indicated the existence of complex wavenumbers. Zemanek's two lowest branches for the case $n = 2$

and the contour-shear and warping branches of the approximate theory together with the lowest torsional branch are plotted in Fig. 1. Only the portions of the frequency spectra for which the wavenumber has positive real and imaginary parts have been plotted in Fig. 1.†

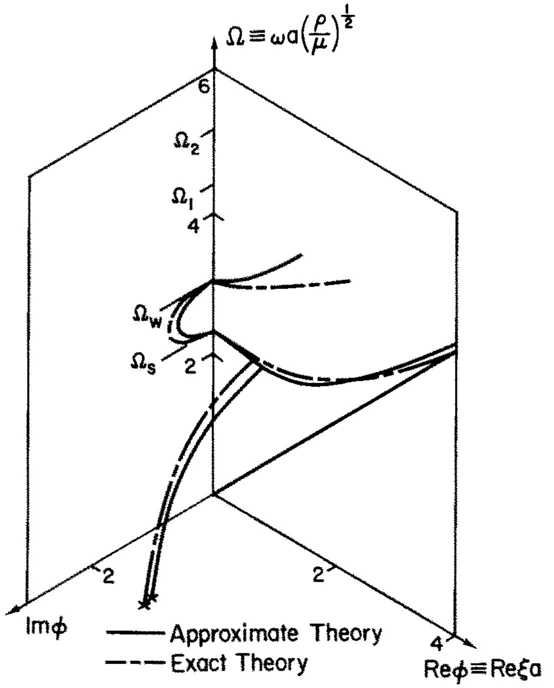

FIG. 1. The three branches of the frequency spectra according to the exact and approximate theories for a circular rod with Poisson's ratio $\nu = 0{\cdot}3$.

Qualitatively the exact and approximate solutions are similar throughout the region illustrated in Fig. 1 and at the cut-off frequencies the two solutions coincide due to the choice of κ_{00} and κ_{22}. The warping cut-off frequency, $\Omega_w = 3{\cdot}0542$ in Fig. 1, is independent of Poisson's ratio whereas the contour-shear cut-off frequency Ω_s decreases with increasing ν and lies in the range $2{\cdot}3362 \leq \Omega_s \leq 2{\cdot}3525$ for ν in the range $0 \leq \nu \leq 0{\cdot}45$. For $\nu = 0{\cdot}3$ the contour-shear cut-off frequency has the value $\Omega_s = 2{\cdot}3479$. The variation with ν for other portions of the curves of Fig. 1 is also quite small.

Both sets of curves possess a minimum in the real Ω–ϕ plane from which emanate two complex branches that extend to zero frequency. The zero frequency intercepts of the two sets of curves are of special interest since in the solutions of static problems of finite rods these complex wavenumbers correspond to displacements confined near the ends. The zero frequency intercepts of (4) are obtained by setting $\Omega = 0$ with the result

$$\phi^2 = -4\kappa_{00} \pm 4\kappa_{00}\left[1 - \frac{3}{\kappa_{00}\kappa_{22}(1+\nu)}\right]^{\frac{1}{2}} \tag{10}$$

† The entire spectra of the three modes being considered is obtained by a mirror reflection of the curves of Fig. 1 in the plane Im $\phi = 0$ followed by another mirror reflection of the resulting curves in the plane Re $\phi = 0$. For a discussion of the ordering of the curves according to the sign of the slope see [7].

There are four roots of this equation which we designate as ϕ_0, $-\phi_0$, $\hat{\phi}_0$ and $-\hat{\phi}_0$ where $\hat{\phi}_0$ is the complex conjugate of ϕ_0.

The zero frequency intercepts of equation (5) are obtained by considering (5) as an implicit function $F_n(\Omega^2, \phi^2) = 0$ and expanding F_n in a Taylor's series in Ω^2 about the plane $\Omega^2 = 0$. Retaining only the first term in Ω^2 in the expansion the requirement that $(\partial F/\partial\Omega^2)]_{\Omega^2=0} = 0$ is obtained. This requirement yields after some manipulation the equation,

$$G_n(\phi) = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = 0 \tag{11}$$

where

$$\begin{aligned} b_{11} &= -2(h^2-1)^{-1}\phi^2 J_n - 2j\phi^3 J_n, \\ b_{12} &= -2\phi^2 J_n'', \\ b_{13} &= 2jn\phi J_n' - 2nJ_n, \\ b_{21} &= n\phi^2 J_n, \\ b_{22} &= -2jn\phi J_n' + 2nJ_n, \\ b_{23} &= \phi^2 J_n'' + j\phi J_n' - n^2 J_n, \\ b_{31} &= 2\phi^2 J_n + \frac{2h^2}{h^2-1} j\phi J_n' + n^2 J_n, \\ b_{32} &= -2j\phi J_n', \\ b_{33} &= -nJ_n, \end{aligned} \tag{12}$$

and where the argument of the Bessel functions is now $j\phi$. For each $n \geq 1$ a distinct characteristic equation results from (11) and for $n = 0$ equation (11) factors into two characteristic equations, one governing the axisymmetric torsional displacements and the other governing the axisymmetric extensional displacements. The characteristic equation for the axisymmetric torsional displacements can be written in the form

$$J_2(j\phi) = 0. \tag{13}$$

This equation is discussed later in connection with the problem of static axisymmetric torsion. For future reference we denote for a given n the mth eigenvalue of equation (11) as $\Phi_{nm} = \alpha_{nm} + j\beta_{nm}$ where α_{nm} and β_{nm} are positive [if Φ_{nm} is a root of (11) then so also are $-\Phi_{nm}$, $\hat{\Phi}_{nm}$ and $-\hat{\Phi}_{nm}$ and thus we can always find such an eigenvalue with positive real and imaginary parts]. Also if for $n = 0$ it is necessary to distinguish between the roots for torsional and extensional displacement, we do so by designating these roots Φ_{0m}^T and Φ_{0m}^E, respectively. Furthermore, we arbitrarily order the eigenvalues so that $\beta_{nm+1} \geq \beta_{nm}$ and thus for a given n, Φ_{n0} is the eigenvalue with the smallest imaginary part. The variations with ν of the root ϕ_0 of (10) with positive real and imaginary parts together with the root Φ_{20} of (11) are sketched in Fig. 2.

Returning to the discussion of the frequency spectra above the plane $\Omega = 0$ we note that above the warping cut-off frequency the exact and approximate solutions begin to

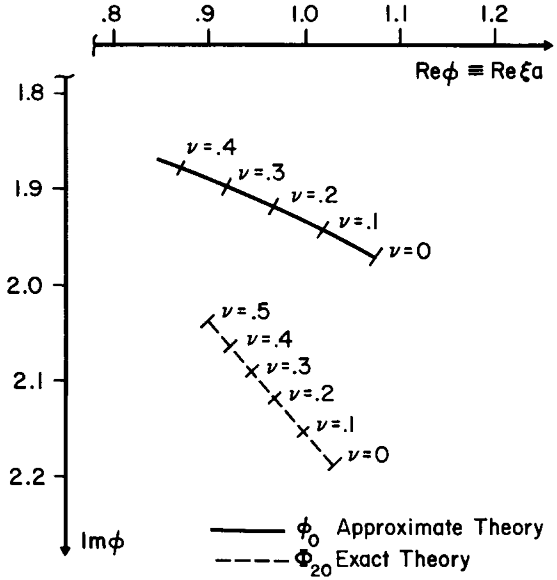

FIG. 2. Zero frequency intercepts of the exact and approximate theories as functions of Poisson's ratio.

diverge due to the proximity of branches of the exact theory which have not been included in the present approximate theory. The next higher branch of equation (5) for $n = 2$ not included in the approximate theory has a cut-off frequency $\Omega_1 = 4{\cdot}3884$ for $\nu = 0{\cdot}3$ which has been indicated in Fig. 1. This branch and the higher branches of (5) for $n = 2$ are coupled to the warping and contour-shear modes in the exact solution, the coupling becoming stronger the higher the frequency. Thus the present approximate theory is limited in application to frequencies less than Ω_1. Since the axisymmetric torsional modes of the exact theory are uncoupled from the non-axisymmetric modes $n = 2$ in an infinite rod and since the lowest axisymmetric mode has been reproduced in the approximate theory, the higher axisymmetric torsional modes do not affect the frequency spectra of Fig. 1. However these modes are of interest in the discussion of the end problem and are considered in Section 5.

3. CORRECTION FACTORS FOR STATIC OR LOW FREQUENCY PROBLEMS

In Section 5 the roots of equation (10) of the approximate theory are employed in the solution of time independent problems. Although the variation between the zero frequency wavenumbers, ϕ_0 and Φ_{20}, is not great it is desirable to have the roots of (10) match exactly the appropriate roots of (11) in order to obtain more accurate results with the approximate theory for time independent problems or for problems at low frequencies. Thus we alter the choice of κ_{00} and κ_{22} so that the two sets of dispersion curves coincide at zero frequency. As Fig. 2 indicates Φ_{20} is complex and since κ_{00} and κ_{22} are yet to be determined we must assume that $1-\{3/[\kappa_{00}\kappa_{22}(1+\nu)]\} < 0$ in order for the roots of (10) to be complex. For convenience we define $\Phi_0 \equiv \Phi_{20}$, $\alpha = \alpha_{20}$ and $\beta \equiv \beta_{20}$ and equating Φ_0^2 to the right hand side of (10) [either the + or − sign may be used for the second term in (10)] we obtain

$$\kappa_{00} = -(\alpha^2-\beta^2)/4 \quad \text{and} \quad \kappa_{22} = \frac{3}{\kappa_{00}(1+\nu)}\left[1+\left(\frac{\alpha\beta}{2\kappa_{00}}\right)^2\right]^{-1} \tag{14}$$

For this choice of static correction factors, the notation Φ_0, $-\Phi_0$, $\hat{\Phi}_0$ and $-\hat{\Phi}_0$ (originally reserved for the zero frequency intercepts of the exact theory) can now be used to indicate the zero frequency intercepts of the approximate theory as determined by equation (10). In Table 1 values of κ_{00} and κ_{22} are tabulated for several values of Poisson's ratio.

TABLE 1. THE CORRECTION FACTORS κ_{00} AND κ_{22} FOR STATIC OR LOW FREQUENCY PROBLEMS OF THE TORSION OF A CIRCULAR ROD

ν	$\Phi_{20} = \Phi_0 = \alpha+j\beta$	κ_{00}	κ_{22}
0·00	1·0327 + j2·1859	0·9279	1·3040
0·05	1·0163 + j2·1681	0·9170	1·2751
0·10	1·0009 + j2·1512	0·9064	1·2482
0·15	0·9863 + j2·1349	0·8962	1·2230
0·20	0·9724 + j2·1193	0·8865	1·1994
0·25	0·9592 + j2·1044	0·8771	1·1774
0·30	0·9466 + j2·0900	0·8680	1·1566
0·35	0·9345 + j2·0762	0·8594	1·1370
0·40	0·9228 + j2·0630	0·8511	1·1186
0·45	0·9116 + j2·0502	0·8431	1·1012
0·50	0·9007 + j2·0380	0·8355	1·0847

4. UNIQUENESS OF SOLUTION

Before proceeding to the discussion of end effects in static torsion a uniqueness theorem is now established which indicates the proper formulation of boundary value problems for the present approximate theory. We consider two solutions of the stress-moment equations (1) and the constitutive equations (2), with each solution consisting of the four stress-moments and the three generalized torsional displacements. A solution is constructed consisting of the differences between the two sets of stress-moments and generalized torsional displacements. Utilizing the stress-moment equations (1) we form the integral,

$$\begin{aligned} 0 = \int_0^t \mathrm{d}t \int_{-l}^{l} & [(T_{31,3}^{(0,1)} - T_{12}^{(0,0)} - \rho I_{02}^{*}\ddot{u}_1^{(0,1)})\dot{u}_1^{(0,1)} \\ & + (T_{32,3}^{(1,0)} - T_{12}^{(0,0)} - \rho I_{20}^{*}\ddot{u}_2^{(1,0)})\dot{u}_2^{(1,0)} \\ & + (T_{33,3}^{(1,1)} - T_{31}^{(0,1)} - T_{32}^{(1,0)} - \rho I_{22}^{*}\ddot{u}_3^{(1,1)})\dot{u}_3^{(1,1)}]\,\mathrm{d}x_3 \end{aligned} \tag{15}$$

where $T_{12}^{(0,0)}$, $T_{31}^{(0,1)}$, $T_{32}^{(1,0)}$, $T_{33}^{(1,1)}$ are now stress-moments of the difference solution and $u_1^{(0,1)}, u_2^{(1,0)}, u_3^{(1,1)}$ are now the generalized torsional displacements of the difference solution. Employing integration by parts and the definitions for kinetic energy density and potential energy density given, respectively, by

$$K \equiv \int_A \tfrac{1}{2}\rho\dot{u}_j\dot{u}_j\,\mathrm{d}x_1\,\mathrm{d}x_2 = \tfrac{1}{2}\rho[I_{02}^{*}\dot{u}_1^{(0,1)^2} + I_{20}^{*}\dot{u}_2^{(1,0)^2} + I_{22}^{*}\dot{u}_3^{(1,1)^2}] \tag{16}$$

and

$$\begin{aligned} U \equiv \int_A \tfrac{1}{2}T_{ij}S_{ij}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \tfrac{1}{2}[& T_{12}^{(0,0)}(u_1^{(0,1)} + u_2^{(1,0)}) \\ & + T_{31}^{(0,1)}(u_{1,3}^{(0,1)} + u_3^{(1,1)}) + T_{32}^{(1,0)}(u_{2,3}^{(1,0)} + u_3^{(1,1)}) + T_{33}^{(1,1)}u_{3,3}^{(1,1)}] \end{aligned} \tag{17}$$

where u_j, T_{ij} and S_{ij} are components of the displacement, stress tensor and strain tensor, respectively, we can write (15) in the form

$$\int_{-l}^{l}[U(t)+K(t)]\,\mathrm{d}x_3=\int_{-l}^{l}[U(0)+K(0)]\,\mathrm{d}x_3 +\int_0^t[T_{31}^{(0,1)}\dot{u}_1^{(0,1)}+T_{32}^{(1,0)}\dot{u}_2^{(1,0)}+T_{33}^{(1,1)}\dot{u}_3^{(1,1)}]_{-l}^{l}\,\mathrm{d}t. \tag{18}$$

For positive definite U and K a solution unique to within a rigid body rotation about the x_3-axis will exist if $\int_{-l}^{l}[U(t)+K(t)]\,\mathrm{d}x_3=0$. Sufficient conditions for this integral to vanish are that in the initial two systems of solutions

$$u_1^{(0,1)},\quad u_2^{(1,0)},\quad u_3^{(1,1)},\quad \dot{u}_1^{(0,1)},\quad \dot{u}_2^{(1,0)},\quad \dot{u}_3^{(1,1)} \tag{19}$$

are specified at $t=0$ for every x_3 and at $x_3=\pm l$ one member of each of the three products

$$T_{31}^{(0,1)}u_1^{(0,1)},\quad T_{32}^{(1,0)}u_2^{(1,0)},\quad T_{33}^{(1,1)}u_3^{(1,1)} \tag{20}$$

is specified for all time t. A uniqueness theorem for time independent problems can be established in an analogous manner with the result that if at $x_3=\pm l$ one member of each of the three products

$$T_{31}^{(0,1)}u_1^{(0,1)},\quad T_{32}^{(1,0)}u_2^{(1,0)},\quad T_{33}^{(1,1)}u_3^{(1,1)} \tag{21}$$

is specified then the solution will be unique to within a rigid body rotation about the x_3-axis. If in fulfilling either of the conditions (20) or (21) the generalized displacements $u_1^{(0,1)}$ and $u_2^{(1,0)}$ are prescribed at $x_3=\pm l$, then the possibility of this superposed rigid body rotation is eliminated.

5. STATIC TORSION OF A CIRCULAR ROD

The problem to be considered consists of a cylindrical rod of length $2l$ that is subjected to a twisting moment T applied at each end. It is assumed that cross sections at $x_3=\pm l$ are free to deform axially and that the cylindrical surfaces are stress free.

The formulation and solution of this problem in the exact three-dimensional theory of elasticity for rods of arbitrary cross section is generally based upon the so-called "semi-inverse method of solution" which was developed by Saint-Venant (see, e.g. [9–11]). In this approach certain assumptions are made concerning the form of the displacements and/or the stresses. Then the equations of equilibrium and the boundary conditions (together with the compatibility equations if the form of stresses is specified) are employed in order to fully determine the unknown variables. For torsion the following displacement field is assumed

$$u_1=-\frac{\theta_0}{l}x_3x_2,\qquad u_2=\frac{\theta_0}{l}x_3x_1,\qquad u_3=\frac{\theta_0}{l}\psi(x_1,x_2) \tag{22}$$

where θ_0/l is the constant twist per unit length and ψ, a function of x_1 and x_2 only, is called the warping function.

From the equations of equilibrium and the boundary conditions on the cylindrical surfaces, it can be shown that ψ is specified by the solution of a problem in potential theory of the Neumann type (see e.g. [10, pp. 109–114]). The twist per unit length can be determined from the solution for ψ together with the total torque T applied at the ends. Once θ_0/l and ψ have been determined the specific distribution of stresses on the bounding surfaces of the rod can be calculated using the solutions (22). Then, according to the uniqueness theorem of the three-dimensional theory of linear elasticity, the solution given by equations (22) is the unique solution corresponding to this particular distribution of stresses at the boundaries.

In a real physical problem it may occur that the total torque T does not arise from this particular distribution of stresses and certainly alternate distributions of stresses can be conceived of which would give rise to the same total torque. When this is the case equation (22) can no longer be assumed to be the unique solution of the torsion problem. In order to handle cases of this nature recourse has been made to the "Saint-Venant principle of equipollent loads" (see [2, 10, 11]). In the present problem this principle states that if the rod is sufficiently long compared to its diameter the solution (22) remains valid in the interior of the rod, no matter how the torque T is applied. That is, the only effects of different distributions of stresses at the ends is to add solutions which are confined near the ends.

The exact determination of these end displacements involves the solutions of boundary value problems in the three-dimensional theory of much greater complexity than the Saint-Venant problem defined above. Explicit solutions of these problems have proven quite difficult to obtain even for the case of a circular cylindrical rod. For example, Purser [3] has solved the problem of static torsion of a circular rod for the case in which the total torque T arises from a general axisymmetric distribution of stresses at the ends [this solution contains the Saint-Venant solution for a circular rod, $\psi(x_1, x_2) = 0$, as a special case]. However, the more general problem for the case in which T arises from a distribution of stresses which is not axisymmetric has apparently not been solved in the three-dimensional theory.

It is now shown that solutions readily available from the approximate theory developed in [1] provide information about these end effects (in particular the end effects in non-axisymmetric torsion) that is not available from the Saint-Venant approach to the torsion problem. In the approximate theory two components of the total torque, namely $T_{32}^{(1,0)}$ and $T_{31}^{(0,1)}$, may be specified at the ends. This allows for a more detailed specification of the applied torque than is possible in the Saint-Venant approach in which only the resultant torque T is specified.

Considerations here are limited to the case of a circular rod although solutions for other cross sections can be determined in a similar manner once the correction factors are determined. Two distributions of the torques $T_{32}^{(1,0)}$ and $T_{31}^{(0,1)}$ at $x_3 = \pm l$ are to be considered, one which is axisymmetric and one which is not, with the requirement that in both cases $T = T_{32}^{(1,0)} - T_{31}^{(0,1)}$. The additional requirement that the ends are free to deform axially leads to the condition that $T_{33}^{(1,1)} = 0$ at $x_3 = \pm l$.

First as in Ref. [1] we transform the equations (1) and (2) in variables $u_1^{(0,1)}$, $u_2^{(1,0)}$ and $u_3^{(1,1)}$ to equations in the variables ψ_1, ψ_2 and ψ_3 where for a circular cylinder the two sets of variables are related by

$$\begin{aligned} \psi_1 &= \tfrac{1}{2}(u_2^{(1,0)} + u_1^{(0,1)}), \\ \psi_2 &= \tfrac{1}{2}(u_2^{(1,0)} - u_1^{(0,1)}), \\ \psi_3 &= r_s u_3^{(1,1)}, \end{aligned} \tag{23}$$

where $r_s = a(8\kappa_{00})^{-\frac{1}{2}}$. For the static torsion of a circular cylinder equations (1) become

$$\begin{aligned} r_s^2\psi_{1,33} - \psi_1 + r_s\psi_{3,3} &= 0, \\ \psi_{2,33} &= 0, \\ 2(1+\nu)r_w^2\psi_{3,33} - \psi_3 - r_s\psi_{1,3} &= 0, \end{aligned} \tag{24}$$

and the constitutive equations (2) become

$$\begin{aligned} T_{12}^{(0,0)} &= \frac{2\mu J_+}{r^2}\psi_1, \\ T_{31}^{(1,0)} &= \frac{\mu J_+}{2r_s}(r_s\psi_{1,3} - r_s\psi_{2,3} + \psi_3), \\ T_{32}^{(0,1)} &= \frac{\mu J_+}{2r_s}(r_s\psi_{1,3} + r_s\psi_{2,3} + \psi_3), \\ T_{33}^{(1,1)} &= \frac{Er_w^2 J_+}{r_s^2}(r_s\psi_{3,3}), \end{aligned} \tag{25}$$

where $r^2 = a^2/2\kappa_{00}$, $r_w^2 = \kappa_{22}a^2/12$ and $J_+ = \pi a^4/2$.

The second of equations (24) is independent of the remaining two equations and can be integrated directly. Furthermore, subtracting the second of equations (25) from the third provides the resultant torque

$$T_{32}^{(0,1)} - T_{31}^{(1,0)} = \mu J_+\psi_{2,3} \tag{26}$$

at a point x_3 in the rod. Thus, whenever the total torque T is specified at the ends of a rod the integral of the second of equations (24) together with (26) yields

$$\psi_2 = \frac{T}{\mu J_+}x_3 \tag{27}$$

to within a rigid body rotation.

Axisymmetric torsion

The first case to be considered is the case in which the total torque is distributed equally between $T_{32}^{(1,0)}$ and $-T_{31}^{(1,0)}$ so that the boundary conditions at $x_3 = \pm l$ are $T_{32}^{(1,0)} = -T_{31}^{(1,1)} = T/2$ and $T_{33}^{(1,1)} = 0$. The solution is obtained by taking ψ_2 to be given by (27) and $\psi_1 = \psi_3 = 0$. The stress-moments as functions of x_3 are then calculated to be

$$T_{12}^{(0,0)} = T_{33}^{(1,1)} = 0 \quad \text{and} \quad T_{32}^{(1,0)} = -T_{31}^{(0,1)} = T/2. \tag{28}$$

The solution of this problem in terms of the variables ψ_1, ψ_2, ψ_3 may be transformed to the variables $u_1^{(0,1)}$, $u_2^{(1,0)}$ and $u_3^{(1,1)}$ by employing the transformation equations (23). Furthermore, the quantity $T/\mu J_+$ can be interpreted as the twist per unit length θ_0/l and the displacement functions corresponding to the above solutions can then be written using the power series expansion of Ref. [1]. The resulting displacements $u_1 = -(\theta_0/l)x_3x_2$, $u_2 = (\theta_0/l)x_3x_1$ and $u_3 = 0$ correspond to the Saint-Venant solution in the three-dimensional theory for the problem of static torsion of a circular rod obtained from (22) by setting $\psi(x_1, x_2) = 0$.

Non-axisymmetric torsion

The second problem to be considered is the non-axisymmetric one in which the total torque is applied only through $T_{32}^{(1,0)}$. In this case the boundary conditions are at $x_3 = \pm l$, $T_{32}^{(1,0)} = T$ and $T_{31}^{(0,1)} = T_{33}^{(1,1)} = 0$. The solution for ψ_2 is again given by (27). We assume ψ_1 and ψ_3 to be of the form $\psi_1 = A_1 e^{j\xi x_3}$ and $\psi_3 = jA_3 e^{j\xi x_3}$, and upon substituting into the first and third of equations (24) obtain

$$\phi^2 = -4\kappa_{00} \pm 4\kappa_{00}\left[1 - \frac{3}{\kappa_{00}\kappa_{22}(1+\nu)}\right]^{\frac{1}{2}}, \tag{29}$$

and

$$A_3 = -A_1 \frac{r_s^2\phi^2 + a^2}{r_s a\phi}, \tag{30}$$

where $\phi = \xi a$. Equation (29) is identical to the equation for the zero frequency intercepts of the approximate theory [equation (10)] and thus, employing the correction factors of Section 3 we can write the roots of (29) as Φ_0, $-\Phi_0$, $\hat{\Phi}_0$ and $-\hat{\Phi}_0$ where $\Phi_0 = \alpha + j\beta$. The general solutions for ψ_1 and ψ_3 take the form

$$\begin{aligned}
\psi_1 &= A_1 e^{j\Phi_0 x_3/a} + B_1 e^{j\hat{\Phi}_0 x_3/a} + C_1 e^{-j\Phi_0 x_3/a} + D_1 e^{-j\hat{\Phi}_0 x_3/a}, \\
\psi_3 &= -\left(\frac{r_s^2\Phi_0^2 + a^2}{r_s a\Phi_0}\right) A_1 e^{j\Phi_0 x_3/a} - \left(\frac{r_s^2\hat{\Phi}_0^2 + a^2}{r_s a\hat{\Phi}_0}\right) B_1 e^{j\hat{\Phi}_0 x_3/a} \\
&\quad + \left(\frac{r_s^2\Phi_0^2 + a^2}{r_s a\Phi_0}\right) C_1 e^{-j\Phi_0 x_3/a} + \left(\frac{r_s^2\hat{\Phi}_0^2 + a^2}{r_s a\hat{\Phi}_0}\right) D_1 e^{-j\hat{\Phi}_0 x_3/a},
\end{aligned} \tag{31}$$

where A_1, B_1, C_1, D_1 are to be determined from the boundary conditions $T_{32}^{(1,0)} = T$ and $T_{33}^{(1,1)} = 0$ at $x_3 = \pm l$. By evaluating the third and fourth of the constitutive equations (25) at the boundaries we obtain four simultaneous algebraic equations which can be solved for the constants A_1, B_1, C_1, D_1 in terms of T, μ, l, a, κ_{00}, κ_{22}, α and β. When these four constants are evaluated in this manner and when the exponential terms are written in terms of sines and cosines and hyperbolic sines and cosines the solutions (31) become

$$\begin{aligned}
\psi_1 &= \frac{Tr_s^2}{\mu a J_+} A\left[2\alpha\beta \cos\frac{\alpha}{a}(l+x_3)\cosh\frac{\beta}{a}(l-x_3)\right. \\
&\quad + (\beta^2-\alpha^2)\sin\frac{\alpha}{a}(l-x_3)\sinh\frac{\beta}{a}(l+x_3) \\
&\quad - 2\alpha\beta\cos\frac{\alpha}{a}(l-x_3)\cosh\frac{\beta}{a}(l+x_3) \\
&\quad \left. - (\beta^2-\alpha^2)\sin\frac{\alpha}{a}(l+x_3)\sinh\frac{\beta}{a}(l-x_3)\right], \\
\psi_3 &= \frac{Tr_s^3}{\mu a^2 J_+}(\alpha^2+\beta^2)A\left[\alpha\cos\frac{\alpha}{a}(l+x_3)\sinh\frac{\beta}{a}(l-x_3)\right. \\
&\quad + \beta\sin\frac{\alpha}{a}(l-x_3)\cosh\frac{\beta}{a}(l+x_3)
\end{aligned} \tag{32}$$

$$+\alpha\cos\frac{\alpha}{a}(l-x_3)\sinh\frac{\beta}{a}(l+x_3)$$

$$+\beta\sin\frac{\alpha}{a}(l+x_3)\cosh\frac{\beta}{a}(l-x_3)\bigg],$$

where $A \equiv [\beta\sin 2(\alpha/a)l-\alpha\sinh 2(\beta/a)l]^{-1}$. The stress moments can be calculated from (25) using the solutions (27) and (32) with the result,

$$\begin{aligned}
T_{12}^{(0,0)} &= \frac{T}{2a}A\bigg[2\alpha\beta\cos\frac{\alpha}{a}(l+x_3)\cosh\frac{\beta}{a}(l-x_3) \\
&\quad +(\beta^2-\alpha^2)\sin\frac{\alpha}{a}(l-x_3)\sinh\frac{\beta}{a}(l+x_3) \\
&\quad -2\alpha\beta\cos\frac{\alpha}{a}(l-x_3)\cosh\frac{\beta}{a}(l+x_3) \\
&\quad -(\beta^2-\alpha^2)\sin\frac{\alpha}{a}(l+x_3)\sinh\frac{\beta}{a}(l-x_3)\bigg], \\
T_{32}^{(1,0)} &= T_{31}^{(0,1)}+T = \frac{T}{2}A\bigg[\beta\sin\frac{\alpha}{a}(l+x_3)\cosh\frac{\beta}{a}(l-x_3) \\
&\quad +\beta\sin\frac{\alpha}{a}(l-x_3)\cosh\frac{\beta}{a}(l+x_3) \\
&\quad -\alpha\cos\frac{\alpha}{a}(l+x_3)\sinh\frac{\beta}{a}(l-x_3) \\
&\quad -\alpha\cos\frac{\alpha}{a}(l-x_3)\sinh\frac{\beta}{a}(l+x_3)\bigg]+\frac{T}{2}, \\
T_{33}^{(1,1)} &= \frac{E}{\mu}\frac{r_w^2r_s^2}{a^3}T(\alpha^2+\beta^2)^2A\bigg[\sin\frac{\alpha}{a}(l-x_3)\sinh\frac{\beta}{a}(l+x_3) \\
&\quad -\sin\frac{\alpha}{a}(l+x_3)\sinh\frac{\beta}{a}(l-x_3)\bigg].
\end{aligned} \tag{33}$$

The solutions (32) and (33) are sketched in Figs. 3 and 4 between the end and a distance of four radii from the end for $\nu = 0{\cdot}3$ and $l/a = 10$. The displacements ψ_1 and ψ_3 of equations (32) approach zero and the stress-moments of equations (33) approach the values given by equations (28) with distance from either end. The solutions of the axi-symmetric and non-axisymmetric problems are indistinguishable on this scale within a distance of about $x_3 = 0{\cdot}3l$ from the end.

The strain energy densities of the non-axisymmetric problem $U(x_3)$ and of the axi-symmetric problem $U_0(x_3)$ are calculated using equations (17) and (23) together with the respective solutions. To illustrate the dependence of the rate of decay in the non-axisymmetric problem on the ratio of the length to the radius of the rod, the quantity $U(x_3)-U_0(x_3)$ has been sketched in Fig. 5 for several ratios of l/a. It can be seen from Fig. 5 that the rate of decay increases as the ratio of l/a increases. The difference in strain energy densities goes to zero relatively faster for larger values of l/a. For shorter rods when l/a is less than or

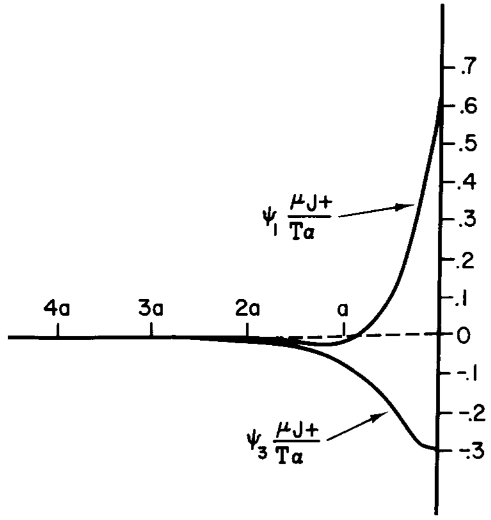

FIG. 3. Displacement solutions for the problem of non-axisymmetric torsion of a circular rod with $l/a = 10$ and $\nu = 0{\cdot}3$.

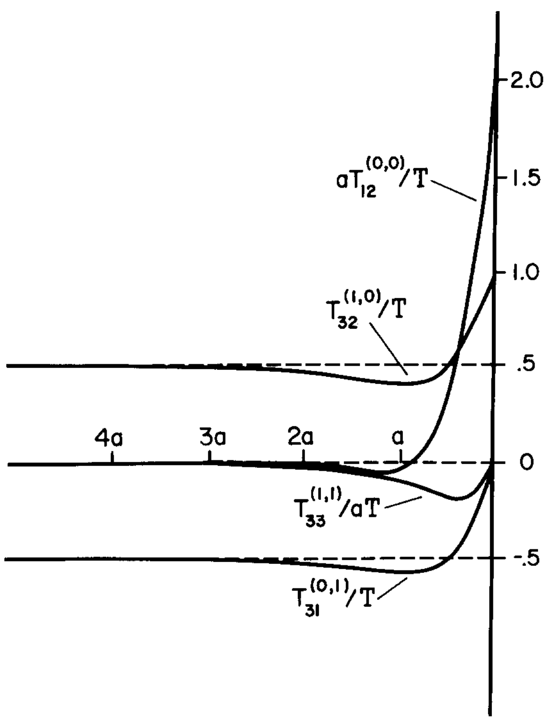

FIG. 4. Stress-moment solutions for the problem of non-axisymmetric torsion of a circular rod with $l/a = 10$ and $\nu = 0{\cdot}3$.

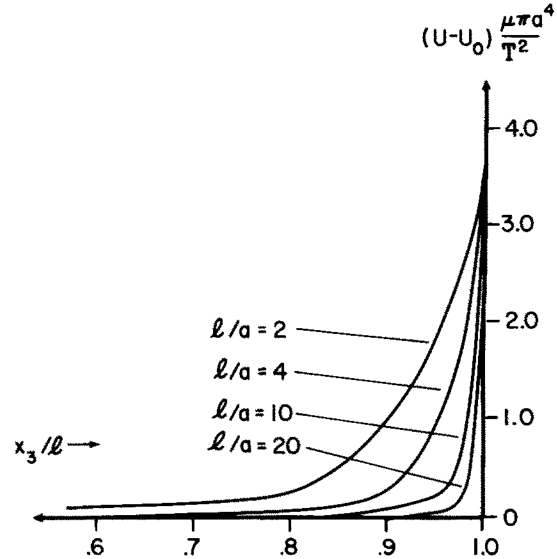

FIG. 5. The difference in strain energy density between the axisymmetric and non-axisymmetric torsion problems for several ratios of l/a and with $\nu = 0{\cdot}3$.

equal to 2, the difference in strain energy densities extends quite far into the interior of the rod before nearing the zero. In fact for $l/a = 2$, this difference does not go to zero, on the scale of Fig. 5, until x_3 is approximately zero. For these values of l/a the effects due to the non-axisymmetric loading can no longer be thought of merely as end effects and the inclusion of them in the solution is essential.

The behavior of the strain energy density, stress-moments and displacements is governed in the solutions (32) and (33) by the exponential terms, sinh and cosh, which decay with distance from either end of the rod. This exponential decay is determined by β, the imaginary part of Φ_0 (we recall that due to the choice of correction factors Φ_0 is the zero frequency wavenumber of the exact solution for the contour-shear and warping modes and, as such, corresponds to a static solution of the exact theory). The smaller β is, the further the "end solutions" extend into the interior of the rod before becoming negligible. It is now shown that the decay of solutions of the approximate theory, as determined by β, actually serves as a minimum for the decay of solutions of the exact three-dimensional theory. Of all the solutions which are available in the three-dimensional theory (this includes, not only higher order axisymmetric and non-axisymmetric torsion solutions, but also end effects due to spurious longitudinal loadings and so forth) the most crucial ones, the ones which decay the slowest, are the ones corresponding to the contour-shear and warping deformations.

Within the framework of the exact three-dimensional theory of linear elasticity exponential solutions which satisfy the condition that the cylindrical surface of the rod is stress free have been found by Dougall [4] (see also [5]). The displacement functions of this solution can be written symbolically in the form

$$u_i^{nm} = \exp(j\Phi_{nm}x_3)f_i^{nm}(x_1, x_2) \tag{34}$$

where the eigenvalues Φ_{nm} are determined by a determinantal equation which is identical to (11). For each eigenvalue Φ_{nm} a solution of the form (34) exists. The decay of the solutions (34) is governed by β_{nm}, the imaginary part of Φ_{nm}, and we refer to these quantities β_{nm} as decay constants. According to Dougall the solutions (34) together with the non-exponential flexure, extension, torsion and bending solutions of Saint-Venant form a complete set of solutions satisfying the homogeneous boundary conditions on the cylindrical surface. It is expected that in the case of a finite rod with a given end loading, the solution should consist of some linear combination of these homogeneous solutions. In practice however, such a linear combination is not easily obtained and the end conditions can be satisfied only approximately in most cases (see, e.g. Lur'e [12]).

One case in which such a linear combination has been obtained is in the problem of axisymmetric torsion which was solved by Purser [3]. Purser considered the problem of a circular rod subject to a general axisymmetric distribution of shearing stresses which gives rise to a resultant torque at the ends. His solution consists of the Saint-Venant solution for torsion plus the axisymmetric solutions of the form (34) corresponding to the eigenvalues Φ_{0m}^T as determined by equation (13). The smallest root of (13) $\Phi_{00}^T = j(5{\cdot}1356)$ serves as a minimum for the decay of the axisymmetric solutions. Support for the use of this value as a minimum for the decay of axisymmetric solutions is found in a paper by Knowles and Sternberg [2]. In their paper Knowles and Sternberg considered the problem of the torsion of a cylinder subject to an axisymmetric self-equilibrating load applied at one end. They proceeded to show that the total strain energy contained between an interior point of the rod and the unloaded end and the stresses decay exponentially away from the loaded end. They established an exponential decay law for the energy and for a circular cylinder determined a decay constant explicitly. This decay constant which serves as a lower bound for the actual decay constant was found by Knowles and Sternberg to be given by the lowest root of equation (13).

TABLE 2. THE ZERO FREQUENCY WAVENUMBERS Φ_{n0} AS DETERMINED FROM THE CHARACTERISTIC EQUATION (11) OF THE EXACT THEORY FOR THE FIRST ELEVEN VALUES OF n

	$\nu = 0{\cdot}0$	$\nu = 0{\cdot}3$	$\nu = 0{\cdot}5$
Φ_{00}^E	$1{\cdot}3890+j2{\cdot}5568$	$1{\cdot}3622+j2{\cdot}7222$	$1{\cdot}3399+j2{\cdot}8106$
Φ_{00}^T	$j5{\cdot}1356$	$j5{\cdot}1356$	$j5{\cdot}1356$
Φ_{10}	$j2{\cdot}6482$	$j2{\cdot}8173$	$j2{\cdot}9016$
Φ_{20}	$1{\cdot}0327+j2{\cdot}1859$	$0{\cdot}9466+j2{\cdot}0900$	$0{\cdot}9007+j2{\cdot}0380$
Φ_{30}	$1{\cdot}2520+j3{\cdot}3921$	$1{\cdot}1414+j3{\cdot}2668$	$1{\cdot}0817+j3{\cdot}1967$
Φ_{40}	$1{\cdot}4136+j4{\cdot}5289$	$1{\cdot}2849+j4{\cdot}3822$	$1{\cdot}2151+j4{\cdot}2986$
Φ_{50}	$1{\cdot}5458+j5{\cdot}6341$	$1{\cdot}4025+j5{\cdot}4703$	$1{\cdot}3242+j5{\cdot}3756$
Φ_{60}	$1{\cdot}6594+j6{\cdot}7210$	$1{\cdot}5036+j6{\cdot}5427$	$1{\cdot}4181+j6{\cdot}4385$
Φ_{70}	$1{\cdot}7597+j7{\cdot}7955$	$1{\cdot}5930+j7{\cdot}6045$	$1{\cdot}5012+j7{\cdot}4920$
Φ_{80}	$1{\cdot}8502+j8{\cdot}8611$	$1{\cdot}6737+j8{\cdot}6590$	$1{\cdot}5762+j8{\cdot}5389$
Φ_{90}	$1{\cdot}9328+j9{\cdot}9201$	$1{\cdot}7475+j9{\cdot}7077$	$1{\cdot}6448+j9{\cdot}5809$
Φ_{100}	$2{\cdot}0092+j10{\cdot}9738$	$1{\cdot}8157+j10{\cdot}7520$	$1{\cdot}7082+j10{\cdot}6189$

The problem of non-axisymmetric torsion of a circular rod has apparently not been solved within the framework of the three-dimensional theory. Indeed this problem is of much greater complexity than that of axisymmetric torsion since in general, depending on the exact distribution of stresses at the ends, any of the solutions (34) for values of n and m

may be necessary to satisfy the boundary conditions. However, an examination of the eigenvalues associated with equation (34) does provide information about the relative importance of the individual solutions u_i^{nm}. As stated previously the rate of decay is governed by the decay constant β_{nm}, the imaginary part of Φ_{nm}. The first root Φ_{n0} for the first eleven values of n has been given in Table 2 for $\nu = 0{\cdot}3$ and also for the limiting cases $\nu = 0{\cdot}0$ and $\nu = 0{\cdot}5$. Throughout this range of ν the decay constant β_{20} is the minimum over the values β_{n0}. For $n > 2$ the values β_{n0} increase with increasing n. Recalling that by definition $\beta \equiv \beta_{20}$ it can be seen that the contour-shear and warping deformations correspond to the solutions of the exact theory with the smallest decay constant. These deformations are of special importance in an end problem, especially one involving torsional loading where they may be excited directly, since they are the ones which decay slowest into the interior.

Acknowledgement—This work was supported in part by the National Science Foundation under a grant to Yale University.

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(*Received* 12 *February* 1971; *revised* 8 *November* 1971)

Абстракт—Сравниваются зависимости дисперсии, полученные из приближенной динамической теории кручения, предложенной в [1] для задачи распространения плоских волн в бесконечном круглом цилиндре, с соответствующими зависимостями точной трехмерной теории и находятся их надлежащая сходимость. Исследуются краевые эффекты статического кручения. Получается решение задачи несимметрического кручения круглого стержня. Оказывается, что скорость уменьшения несимметрических краевых эффектов меньша такой же в аналогических осесимметрических краевых эффектах.